Homework #4
Quantum Mechanics

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Problem #2.24

If a beam of spin-2/3 particles is input to a Stern Gerlach analyzer, there are four possible output beams whose deflections are consistant with magnetic moments arising from spin angular momentum components $\frac{3}{2}\hbar, \frac{1}{2}\hbar, -\frac{1}{2}\hbar$, and $-\frac{3}{2}\hbar$. For a spin-3/2 system:

a) Write down the eigenvalue equations for the S_z operator
b) Write down the matrix representation of the S_z eigenstates.
c) Write down the matrix representation of the S_z operator.
d) Write down the eigenvalue equations for the $\mathbf{S}^2$ operator.
e) Write down the matrix representation of the $\mathbf{S}^2$ operator.

<u>Sln:</u>

a)

The three eigenvalue equations for the S_z operator are given by

$$\begin{aligned} S_z \left|2\right\rangle &= \frac{3}{2}\hbar \left|2\right\rangle \\ S_z \left|1\right\rangle &= \frac{1}{2}\hbar \left|1\right\rangle \\ S_z \left|-1\right\rangle &= -\frac{1}{2}\hbar \left|-1\right\rangle \\ S_z \left|-2\right\rangle &= -\frac{3}{2}\hbar \left|-2\right\rangle \end{aligned}$$

Where the three states have been labled $\left|2\right\rangle, \left|1\right\rangle, \left|-1\right\rangle$, and $\left|-2\right\rangle$

b)

Since all these states must be orthogonal, we can represent them as,

$$|2\rangle \doteq \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |1\rangle \doteq \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |-1\rangle \doteq \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |-2\rangle \doteq \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

c)

Since S_z is represented by a square matrix, by looking at the equations determined by parts **a)** and **b)**, we know S_z must be a 4×4 matrix. So we can say,

$$S_z \doteq \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Further, since we are representing this system in its own basis set, the operator S_z can be represented by a diagonal matrix (meaning $a_{ij} = 0$ if $i \neq j$) and the diagonal components of this matrix will be the eigenvalues (meaning $a_{11} = \lambda_1, a_{22} = \lambda_2, a_{33} = \lambda_3$, and $a_{44} = \lambda_4$). Therefore,

$$S_z \doteq \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

d)

In general, the $\mathbf{S}^2$ operator can be represented by

$$\mathbf{s}^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle \tag{1}$$

where s represents the spin of the system, and m represents the eigenvalue divided by $\hbar$ (this is so m comes out to be either an integer or half integer for simplicity reasons). In this instance, $s = \frac{3}{2}$ and $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$and $-\frac{3}{2}$.We then have the equations

$$\mathbf{S}^2 \left|\frac{3}{2}\frac{3}{2}\right\rangle = \frac{15}{4}\hbar^2 \left|\frac{3}{2}\frac{3}{2}\right\rangle$$
$$\mathbf{S}^2 \left|\frac{3}{2}\frac{1}{2}\right\rangle = \frac{15}{4}\hbar^2 \left|\frac{3}{2}\frac{1}{2}\right\rangle$$
$$\mathbf{S}^2 \left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{15}{4}\hbar^2 \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$
$$\mathbf{S}^2 \left|\frac{3}{2}, -\frac{3}{2}\right\rangle = \frac{15}{4}\hbar^2 \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

or, using the more familiar notation used throuought the rest of this problem,

$$
\begin{aligned}
\mathbf{S}^2 |2\rangle &= \frac{15}{4}\hbar^2 |2\rangle \\
\mathbf{S}^2 |1\rangle &= \frac{15}{4}\hbar^2 |1\rangle \\
\mathbf{S}^2 |{-1}\rangle &= \frac{15}{4}\hbar^2 |{-1}\rangle \\
\mathbf{S}^2 |{-1}\rangle &= \frac{15}{4}\hbar^2 |{-2}\rangle
\end{aligned}
$$

e)

Looking at the abouve situation, it can be seen that $\mathbf{S}^2$ must be represented by a 4×4 matrix. Therefore $\mathbf{S}^2$ has the form

$$
\mathbf{S}^2 \doteq \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
$$

From the equations in **d)**, we get the following:

From the equation for $|2\rangle$:

$$
\mathbf{S}^2 |2\rangle = \frac{15}{4}\hbar^2 |2\rangle \implies \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\implies \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} a_{11} = \frac{15}{4}\hbar^2 \\ a_{21} = 0 \\ a_{31} = 0 \\ a_{41} = 0 \end{cases}
$$

From the equation for $|1\rangle$:

$$
\mathbf{S}^2 |1\rangle = \frac{15}{4}\hbar^2 |1\rangle \implies \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\implies \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} a_{12} = 0 \\ a_{22} = \frac{15}{4}\hbar^2 \\ a_{32} = 0 \\ a_{42} = 0 \end{cases}
$$

From the equation for $|{-1}\rangle$:

$$\mathbf{S}^2 |{-1}\rangle = \frac{15}{4}\hbar^2 |{-1}\rangle \implies \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \implies \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = \frac{15}{4}\hbar^2 \\ a_{43} = 0 \end{cases}$$

and finaly, from the equation for $|{-2}\rangle$:

$$\mathbf{S}^2 |{-2}\rangle = \frac{15}{4}\hbar^2 |{-2}\rangle \implies \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{pmatrix} = \frac{15}{4}\hbar^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \implies \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = 0 \\ a_{43} = \frac{15}{4}\hbar^2 \end{cases}$$

Therefore, the $\mathbf{S}^2$ matrix representation can be written as,

$$\mathbf{S}^2 \doteq \frac{15}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem #2.25

Are the projection operators P_+ and P_- Hermitian? Explain.

<u>Sln:</u>

$$P_+ = |+\rangle\langle +| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_- = |-\rangle\langle -| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For a matrix, A, to be Hermitian, $A^\dagger = A$.

$$P_+^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_+$$
$$P_-^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = P_-$$

Therefore, yes P_+ and P_- are Hermitian.[1]

Problem #3.2

Show that the probability of a measurement of the energy is time independent for a general state $|\psi(t)\rangle = \sum_n c_n(t)\,|E_n\rangle$ that evolves due to a time-independent Hamiltonian. Show that the probability of measurements of other observables are also time independent if those observables commute with the Hamiltonian.

<u>Sln:</u>

Since the Hamiltonian is time independent, the time evolution of the state $|\psi\rangle$ can be given by

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}\,|E_n\rangle$$

The probability, $\mathcal{P}$, of measuring of measuring $|\psi\rangle$ as having a particular energy, E_n, at a particulat time t, can be given by

$$\begin{aligned}
\mathcal{P}_n(t) &= |\,\langle E_n|\psi(t)\rangle\,|^2 \\
&= |\,\langle E_n|\,(\sum_n c_n e^{-iE_n t/\hbar}\,|E_n\rangle)|^2 \\
&= |c_n e^{-iE_n t/\hbar}|^2 \\
&= (c_n^*)(c_n)(e^{iE_n t/\hbar})(e^{-iE_n t/\hbar}) \\
&= |c_n|^2 e^0 \\
&= |c_n|^2
\end{aligned}$$

Further, the probability of measuring a particular value, a_i for a generic observable A, that commutes with the Hamiltonian is,

$$\mathcal{P}_{a_i} = |\,\langle a_i|\psi(t)\rangle\,|^2$$

Also since $\sum_n |E_n\rangle$ forms a compleat bsis set,

$$|a_i\rangle = \sum_n b_n e^{-iE_n t/\hbar}\,|E_n\rangle$$

[1]The $\dagger$ notation represents both a transpose and complect conjuget transformation

We know that A and H commute. Therefore they have the same eigenvectors.[2] We know the eigenvectors of H are $E_n|_1^{all}$. Thus the eigenvectors of A must also be $E_n|_1^{all}$. This means a_i must be some E_i.

$$
\begin{aligned}
\mathcal{P}_{a_i} &= |\left\langle a_i|\psi(t)\right\rangle|^2 \\
&= |\left\langle E_i\right|(\sum_n^{all} c_n e^{-iE_n t/\hbar}\left|E_n\right\rangle)|^2 \\
&= |\sum_n^{all} c_n e^{-iE_n t/\hbar}\left\langle E_i|E_n\right\rangle|^2 \\
&= |c_i e^{-iE_i t/\hbar}|^2 \\
&= |c_i|^2
\end{aligned}
$$

Therefore, for any observable that commutes with the Hamiltonian, is time independent.

Problem #3.13

Let the matrix representation of the Hamiltonian of a three-state system be

$$
H = \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}
$$

using the basis states $\left|1\right\rangle, \left|2\right\rangle$, and $\left|3\right\rangle$

a) If the state of the system at time $t=0$ is $\left|\psi(0)\right\rangle = \left|2\right\rangle$, what is the probability that the system is in state $\left|2\right\rangle$ at time t?

b) If, instead, the state of the system at time $t=0$ is $\left|\psi(0)\right\rangle = \left|3\right\rangle$, what is the probability that the system is in state $\left|3\right\rangle$ at time$t$?

<u>Sln:</u>

The Hamiltonian is

$$
H\left|E_n\right\rangle = E_n\left|E_n\right\rangle
$$

[2] I have not found a way to simply prove this

Lets diagonalize the Hamiltonian to determine the allowed values for E_n

$$det(H - \lambda I) = \begin{vmatrix} E_0 - \lambda & 0 & A \\ 0 & E_1 - \lambda & 0 \\ A & 0 & E_0 - \lambda \end{vmatrix} = 0$$

$$(E_0 - \lambda)(E_1 - \lambda)(E_0 - \lambda) + A(-A)(E_1 - \lambda) = 0$$

$$(E_1 - \lambda)[(E_0 - \lambda)^2 - A^2] = 0$$

$$\begin{cases} E_1 - \lambda = 0 & \implies \lambda = E_1 \\ E_0 - \lambda + A = 0 & \implies \lambda = E_0 + A \\ E_0 - \lambda - A = 0 & \implies \lambda = E_0 - A \end{cases}$$

Now, this will be used to determine the eigenvectors (Which are the energy vectors)

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = E_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{cases} E_0 x_1 + A z_1 = E_1 x_1 \\ E_1 y_1 = E_1 y_1 \\ A x_1 + E_0 z_1 = E_1 z_1 \end{cases}$$

$$|E_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = (E_0 + A) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\begin{cases} E_0 x_2 + A z_2 = (E_0 + A) x_2 \\ E_1 y_2 = (E_0 + A) y_2 \\ A x_2 + E_0 z_2 = (E_0 + A) z_2 \end{cases}$$

$$|E_2\rangle = \text{Norm} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = (E_0 - A) \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

$$\begin{cases} E_0 x_3 + A z_3 = (E_0 - A) x_3 \\ E_1 y_3 = (E_0 - A) y_3 \\ A x_3 + E_0 z_3 = (E_0 - A) z_3 \end{cases}$$

$$|E_3\rangle = \text{Norm} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

From this we can determine the relationship between the basis states and the energy states.

$$\begin{aligned} |E_1\rangle &= |2\rangle \\ |E_2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle) \\ |E_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle) \end{aligned}$$

and finaly

$$\begin{aligned} |1\rangle &= \frac{1}{\sqrt{2}}(|E_2\rangle + |E_3\rangle) \\ |2\rangle &= |E_1\rangle \\ |3\rangle &= \frac{1}{\sqrt{2}}(|E_2\rangle - |E_3\rangle) \end{aligned}$$

Now that thats out of the way, lets get started on the problems!

a)

At $t = 0$

$$|\psi(0)\rangle = |2\rangle = |E_1\rangle$$

From here we can look at the time evolution

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar} |E_1\rangle$$

Now the probability of measureing $|\psi(t)\rangle$ of being $|2\rangle$ is

$$\begin{aligned} \mathcal{P}_2 &= | \langle 2|\psi(t)\rangle |^2 \\ &= | \langle E_1| (e^{-iE_1 t/\hbar} |E_1\rangle)|^2 \\ &= |(e^{-iE_1 t/\hbar} \langle E_1|E_1\rangle)|^2 \\ &= |(e^{-iE_1 t/\hbar}|^2 \\ &= 1 \end{aligned}$$

This is independent of time.

b)

For $|\psi(0)\rangle = |3\rangle$, we have

$$|\psi(0)\rangle = |3\rangle = \frac{1}{\sqrt{2}}(|E_2\rangle - |E_3\rangle)$$

and the time evolution

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-iE_2t/\hbar}|E_2\rangle - e^{-iE_3t/\hbar}|E_3\rangle)$$

The probability of measuring $|3\rangle$ at time t is

$$\begin{aligned}
\mathcal{P}_3 &= |\langle 3|\psi(t)\rangle|^2 \\
&= |(\langle E_2|\frac{1}{\sqrt{2}} - \langle E_3|\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}|E_2\rangle - \frac{1}{\sqrt{2}}e^{-iE_3t/\hbar}|E_3\rangle)|^2 \\
&= |\frac{1}{2}e^{-iE_2t/\hbar} + \frac{1}{2}e^{-iE_3t/\hbar}|^2 \\
&= \frac{1}{4}|e^{-iE_2t/\hbar} + e^{-iE_3t/\hbar}|^2
\end{aligned}$$

Since The energy of $|E_2\rangle = E_0 + A$ and the energy of $|E_3\rangle = E_0 + A\ldots$

$$\begin{aligned}
&= \frac{1}{4}\left|e^{-i(E_0+A)t/\hbar} + e^{-i(E_0-A)t/\hbar}\right|^2 \\
&= \frac{1}{4}\left|e^{-iE_0t/\hbar}\left(e^{-iAt/\hbar} + e^{iAt/\hbar}\right)\right|^2 \\
&= \frac{1}{2}\left|e^{-iE_0t/\hbar}\right|^2\left|2\cosh\left(\frac{-iAt}{\hbar}\right)\right|^2 \\
&= \left|\cos\left(-\frac{At}{\hbar}\right)\right| \\
&= \cos^2\left(\frac{At}{\hbar}\right)
\end{aligned}$$

It can be seen that this is not independent of time.

Problem #3.15

Show that the general energy state superposition $|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$ satisfies the Schrödinger equation, but not energy eigenvalue equation.

Sln:

[3] The Scrödinger Equation...

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= H(t)|\psi(t)\rangle \\
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle &= H(t)\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle \\
i\hbar\sum_n c_n\frac{-i}{\hbar}e^{-iE_nt/\hbar}E_n|E_n\rangle &= \sum_n c_n e^{-iE_nt/\hbar}H(t)|E_n\rangle \\
\sum_n c_n e^{-iE_nt/\hbar}E_n|E_n\rangle &= \sum_n c_n e^{-iE_nt/\hbar}E_n|E_n\rangle
\end{aligned}$$

Now to show $|\psi(t)\rangle$ is not an energy eigenvalue. Presenting...the energy eigenvalue equation!

$$H|\psi\rangle = E|\psi\rangle$$

where E is the eigenvalue of ψ

$$\begin{aligned}
H\left(\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle\right) &= E\left(\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle\right) \\
\sum_n c_n e^{-iE_nt/\hbar}H|E_n\rangle) &= \sum_n c_n e^{-iE_nt/\hbar}E|E_n\rangle \\
\sum_n c_n e^{-iE_nt/\hbar}E_n|E_n\rangle) &= \sum_n c_n e^{-iE_nt/\hbar}E|E_n\rangle .
\end{aligned}$$

This implies $E + E_1, E = E_2, \ldots, E = E_n$ which itself implies $E_1 = E_2 = \cdots = E_n$. Therefore the only way $|\psi\rangle$ can satisfy the energy eigenvalue equation is if it is only represented by one energy eigenstate (and is therefor that energy eigenstate itself). Therefore, this, in general, does not satify the energy eigenvalue equation.

Problem #3.16

For a spin-1/2 system undergoing Rabi oscillations, (i.e., Rabi flopping) assume that the resonance condition $\omega = \omega_0$ holds.

[3] this problem is unfinished as I do not understand it. Specifically, how to compare the H term.

a) Solve the differential equations for the coefficients $\alpha_\pm(t)$. Use your results to find the transformed state vector $\left|\widetilde{\psi}(t)\right\rangle$ and the state vector $|\psi(t)\rangle$, assuming the most general initial state of the system.

b) Verify that a π-pluse ($\omega_1 t = \pi$)produces a spin flip (i.e., $\mathcal{P}_+(t) = \mathcal{P}_-(0)$ and $\mathcal{P}_-(t) = \mathcal{P}_+(0)$). Calculate both the transformed state vector $\left|\widetilde{\psi}(t)\right\rangle$ and the state vector $|\psi(t)\rangle$.

c) Assume that the interaction time is such that $\omega_1 t = \pi/2$. Find the effect on the syaytem if the initial state is $|+\rangle$.

d) Discuss the difference between the original reference frame and the rotating reference frame in light of your results.

Sln:

(a)

The differential equations for $\alpha_\pm$ are given in the book (p.89 eq.(3.94))

$$\begin{aligned} i\hbar\dot{\alpha}_+(t) &= -\frac{\hbar\Delta\omega}{2}\alpha_+(t) + \frac{\hbar\omega_1}{2}\alpha_-(t) \\ i\hbar\dot{\alpha}_-(t) &= \frac{\hbar\omega_1}{2}\alpha_+(t) + \frac{\hbar\Delta\omega}{2}\alpha_-(t) \end{aligned}$$

Where $\Delta\omega = \omega - \omega_0$. Since one of the conditions imposed is$\omega = \omega_0$, we then know that $\Delta\omega = 0$. We then have,

$$\begin{aligned} i\hbar\dot{\alpha}_+(t) &= \frac{\hbar\omega_1}{2}\alpha_-(t) & i\hbar\dot{\alpha}_-(t) &= \frac{\hbar\omega_1}{2}\alpha_+(t) \\ i\hbar\dot{\alpha}_+(t) - \frac{\hbar\omega_1}{2}\alpha_-(t) &= 0 & i\hbar\dot{\alpha}_-(t) - \frac{\hbar\omega_1}{2}\alpha_+(t) &= 0 \\ i\dot{\alpha}_+(t) - \frac{\omega_1}{2}\alpha_-(t) &= 0 & i\dot{\alpha}_-(t)\frac{\omega_1}{2}\alpha_+(t) &= 0 \end{aligned}$$

These equations are satisfied if

$$\alpha_+(t) = Ae^{-i\omega_1 t/2} + Be^{i\omega_1 t/2} \qquad \alpha_-(t) = Ae^{-1\omega_1 t/2} - Be^{i\omega_1 t/2}$$

Lets check to make sure this is true[4]

$$\begin{aligned} i\dot{\alpha}_+(t) - \frac{\omega_1}{2}\alpha_-(t) &= 0 \\ i\Big[-Ai\frac{\omega_1}{2}e^{-i|omega_1 t/2} + Bi\frac{\omega_1}{2}e^{i\omega_1 t/2}\Big] - \frac{\omega_1}{2}\Big[Ae^{-1\omega_1 t/2} - Be^{i\omega_1 t/2}\Big] &= 0 \\ A\frac{\omega_1}{2}e^{-i|omega_1 t/2} - B\frac{\omega_1}{2}e^{i\omega_1 t/2} - A\frac{\omega_1}{2}e^{-1\omega_1 t/2} - B\frac{\omega_1}{2}e^{i\omega_1 t/2} &= 0 \\ 0 &= 0 \end{aligned}$$

[4] This is the 8^{th} set of equations I have done this check for. Needless to say, equations 1-5 did not work.

$$i\dot{\alpha}_-(t) + \frac{\omega_1}{2}\alpha_+(t) = 0$$
$$i\Big[-Ai\frac{\omega_1}{2}e^{-i|omega_1 t/2} - Bi\frac{\omega_1}{2}e^{i\omega_1 t/2}\Big] - \frac{\omega_1}{2}\Big[Ae^{-1\omega_1 t/2} + Be^{i\omega_1 t/2}\Big] = 0$$
$$A\frac{\omega_1}{2}e^{-i|omega_1 t/2} + B\frac{\omega_1}{2}e^{i\omega_1 t/2} - A\frac{\omega_1}{2}e^{-1\omega_1 t/2} + B\frac{\omega_1}{2}e^{i\omega_1 t/2} = 0$$
$$0 = 0$$

Therefore,

$$\left|\widetilde{\psi}(t)\right\rangle \doteq \begin{pmatrix}\alpha_+(t)\\ \alpha_-(t)\end{pmatrix} = \begin{pmatrix}Ae^{-i\omega_1 t/2} + Be^{i\omega_1 t/2}\\ Ae^{-1\omega_1 t/2} - Be^{i\omega_1 t/2}\end{pmatrix}$$

We know from the book (pg.89 eq.(3.94)), that $|\psi\rangle$ can be written as

$$|\psi\rangle \doteq \begin{pmatrix}\alpha_+(t)e^{-i\omega t/2}\\ \alpha_-(t)e^{i\omega t/2}\end{pmatrix}$$
$$= \begin{pmatrix}\big(Ae^{-i\omega_1 t/2} + Be^{i\omega_1 t/2}\big)e^{-i\omega_0 t/2}\\ \big(Ae^{-i\omega_1 t/2} - Be^{i\omega_1 t/2}\big)e^{i\omega_0 t/2}\end{pmatrix}$$

In order to say anything about probabilities, we[5] must make sure this vector is normalized/

$$
\begin{aligned}
1 &= \langle\psi|\psi\rangle \\
&= \bigg[C^*\Big(A^* e^{i\omega_1 t/2} + B^* e^{-i\omega_1 t/2}\Big)e^{i\omega_0 t/2}\langle+| \\
&\qquad + C^*\Big(A^* e^{i\omega_1 t/2} - B^* e^{-i\omega_1 t/2}\Big)e^{-i\omega_0 t/2}\langle-|\bigg] \\
&\qquad \bigg[C\Big(A e^{-i\omega_1 t/2} + B e^{i\omega_1 t/2}\Big)e^{-i\omega_0 t/2}|+\rangle \\
&\qquad + C\Big(A e^{-i\omega_1 t/2} - B e^{i\omega_1 t/2}\Big)e^{i\omega_0 t/2}|-\rangle\bigg] \\
&= |C|^2\bigg[|A|^2 + |B|^2 + A^*Be^{2i\omega_1 t/2} + AB^*e^{2\omega_1 t/2} \\
&\qquad + |A|^2 + |B|^2 - A^*Be^{2i\omega_1 t/2} - AB^*e^{2\omega_1 t/2}\bigg] \\
&= |C|^2 \cdot 2\Big(|A|^2 + |B|^2\Big) \\
\implies |C|^2 &= \frac{1}{2\Big(|A|^2 + |B|^2\Big)}
\end{aligned}
$$

lets take C to be the real solution

$$C = \frac{1}{\sqrt{2\Big(|A|^2 + |B|^2\Big)}}$$

We can then express $|\psi\rangle$ as

$$|\psi(t)\rangle \doteq \frac{1}{\sqrt{2\Big(|A|^2 + |B|^2\Big)}}\begin{pmatrix}\big(Ae^{-i\omega_1 t/2} + Be^{i\omega_1 t/2}\big)e^{-i\omega_0 t/2} \\ \big(Ae^{-i\omega_1 t/2} - Be^{i\omega_1 t/2}\big)e^{i\omega_0 t/2}\end{pmatrix}$$

It can be seen that there must be some relationship between A and B. To look at this, lets let

$$A = r_1 e^{\theta_1} \qquad B = r_2 e^{i\theta_2}$$

[5] It seems to sound better to me to write as if I were multiple people. "I must make sure. . . " or "Now I am now going to. . . " just sounds wrong in many instances.

This makes the values

$$\frac{A}{\sqrt{|A|^2+|B|^2}} \qquad \frac{B}{\sqrt{|A|^2+|B|^2}}$$
$$\frac{r_1}{\sqrt{|r_1|^2+|r_2|^2}} \cdot e^{i\theta_1} \qquad \frac{r_2}{\sqrt{|r_1|^2+|r_2|^2}} \cdot e^{i\theta_2}$$

Lets let ϕ represent the relationship between the magnitudes of A and B so that

$$\frac{A}{\sqrt{|A|^2+|B|^2}} = \sin\phi \qquad \frac{B}{\sqrt{|A|^2+|B|^2}} = \cos\phi$$

With this, the state function can be written as

$$|\psi(t)\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} \left(\sin(\phi)e^{i\theta_1}e^{-i\omega_1 t/2} + \cos(\phi)e^{i\theta_2}e^{i\omega_1 t/2}\right)e^{-i\omega_0 t/2} \\ \left(\sin(\phi)e^{i\theta_1}e^{-i\omega_1 t/2} - \cos(\phi)e^{i\theta_2}e^{i\omega_1 t/2}\right)e^{i\omega_0 t/2} \end{pmatrix}$$

The Probability function To simplify things, $\mathcal{P}_+(t)$ and $\mathcal{P}_-(t)$ will be derived.

$$\begin{aligned}
\mathcal{P}_+ &= |\langle +|\psi(t)\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\sin(\phi)e^{i\theta_1}e^{-i\omega_1 t/2} + \cos(\phi)e^{i\theta_2}e^{i\omega_1 t/2}\right)e^{-i\omega_0 t/2}\right|^2 \\
&= \frac{1}{2}\left|e^{-i\omega_0 t/2}\right|^2 \cdot \left(\sin(\phi)e^{-i\theta_1}e^{i\omega_1 t/2} + \cos(\phi)e^{-i\theta_2}e^{-i\omega_1 t/2}\right)\left(\sin(\phi)e^{i\theta_1}e^{-i\omega_1 t/2} + \cos(\phi)e^{i\theta_2}e^{i\omega_1 t/2}\right) \\
&= \frac{1}{2}\left(\sin^2(\phi) + \cos^2(\phi) + \sin(\phi)\cos(\phi)e^{-i[(\theta_1-\theta_2)-(\omega_1 t/2+\omega_1 t/2)]} + \sin(\phi)\cos(\phi)e^{i[(\theta_1-\theta_2)-(\omega_1 t/2+\omega_1 t/2)]}\right) \\
&= \frac{1}{2}\left[1 + \sin(\phi)\cos(\phi)\left(e^{i[(\theta_1-\theta_2)-(\omega_1 t/2+\omega_1 t/2)]} + e^{-i[(\theta_1-\theta_2)-(\omega_1 t/2+\omega_1 t/2)]}\right)\right] \\
&= \frac{1}{2}[1 + 2\sin(\phi)\cos(\phi)\cosh(i[(\theta_1-\theta_2)-(\omega_1 t/2+\omega_1 t/2)])] \\
&= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos[(\theta_1-\theta_2)-(\omega_1 t)] \\
&= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos(\Delta\theta - \omega_1 t) \\
\mathcal{P}_- &= 1 - \mathcal{P}_+ \\
&= \frac{1}{2} - \sin(\phi)\cos(\phi)\cos(\Delta\theta - \omega_1 t)
\end{aligned}$$

where $\Delta\theta = \theta_1 - \theta_2$. This will also hold true for $\left|\widetilde{\psi}(t)\right\rangle$ as the only difference will be the $e^{\pm i\omega_0 t/2}$ term which can be pulled out and delt with seperatly. Once you pull this term out, take the absolute value and square it, it just ends up being one and not effecting the overall probability.

b):

First, what is $\mathcal{P}_+(0)$ and $\mathcal{P}_-(0)$?

$$\mathcal{P}_+(0) = \frac{1}{2} + \sin(\phi)\cos(\phi)\cos(\Delta\theta)\mathcal{P}_-(0) \qquad = \frac{1}{2} - \sin(\phi)\cos(\phi)\cos(\Delta\theta)$$

Now, what are $\mathcal{P}_+(\frac{\pi}{\omega_1})$ and $\mathcal{P}_-(\frac{\pi}{\omega_1})$?

$$\begin{aligned}
\mathcal{P}_+\left(\frac{\pi}{\omega_1}\right) &= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos\left(\Delta\theta - \omega_1\frac{\pi}{\omega_1}\right) \\
&= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos(\Delta\theta - \pi) \\
&= \frac{1}{2} - \sin(\phi)\cos(\phi)\cos(\Delta\theta) \\
\mathcal{P}_-\left(\frac{\pi}{\omega_1}\right) &= \frac{1}{2} - \sin(\phi)\cos(\phi)\cos\left(\Delta\theta - \omega_1\frac{\pi}{\omega_1}\right) \\
&= \frac{1}{2} - \sin(\phi)\cos(\phi)\cos(\Delta\theta - \pi) \\
&= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos(\Delta\theta)
\end{aligned}$$

Yes, this does the thing it is supposed to do.[6]

c)

Since the state starts in $|+\rangle$, $\phi = \pi/4$ and $\Delta\theta = 0$. Lets have $\theta_1 = 0$ and $\theta_2 = 0$. We then have

$$|\psi(t)\rangle \doteq \frac{1}{2}\begin{pmatrix} \left(e^{-i\omega_1 t/2} + e^{i\omega_1 t/2}\right)e^{-i\omega_0 t/2} \\ \left(e^{-i\omega_1 t/2} - e^{i\omega_1 t/2}\right)e^{i\omega_0 t/2} \end{pmatrix}$$

Lets go back to out$\mathcal{P}_+(t)$ and $\mathcal{P}_-(t)$ equations.

$$\begin{aligned}
\mathcal{P}_+\left(\frac{\pi}{2\omega_1}\right) &= \frac{1}{2} + \sin(\phi)\cos(\phi)\cos\left(\Delta\theta - \omega_1\left(\frac{\pi}{2\omega_1}\right)\right) \\
&= \frac{1}{2} + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\cos\left(0 - \frac{\pi}{2}\right) \\
&= \frac{1}{2} + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{\pi}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2}(0) \\
&= \frac{1}{2}
\end{aligned}$$

[6] this is usually the time in the problem where I find I had made a mistake as this diddn't do the thing it was supposed to do many times

$$
\begin{aligned}
\mathcal{P}_-\left(\frac{\pi}{2\omega}\right) &= \frac{1}{2} - \sin(\phi)\cos(\phi)\cos\left(\Delta\theta - \omega_1\left(\frac{\pi}{2\omega_1}\right)\right) \\
&= \frac{1}{2} - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\cos\left(0 - \frac{\pi}{2}\right) \\
&= \frac{1}{2} - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{\pi}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2}(0) \\
&= \frac{1}{2}
\end{aligned}
$$

d)

As previously discussed, the differences in the probabilities considering $|\psi(t)\rangle$ and $\left|\widetilde{\psi}(t)\right\rangle$ are not apparent. However, in terms of solving the differential equations for their coefficients, it is much easier to consider $\left|\widetilde{\psi}(t)\right\rangle$ than it is to consider $|\psi(t)\rangle$.

Problenm #3.17

Consider an electron neutrino with an energy of $8MeV$. How far must this neutrino travel befor it oscillates to a muon neutrino? Assume the neutrino mixing parameters given in the text. How many complete oscillations ($v_e \rightarrow v_\mu \rightarrow v_e$) will take place if this neutrino travels from the sun to the earth? Through the earth?

<u>Sln:</u>

From equation (3.79) in the book, we know that the probability an electron neutrino will be detected to be a muon neutrino is

$$\mathcal{P}_{v_e \rightarrow v_\mu} = \sin^2(\theta)\sin^2\left(\frac{(m_1^2 - m_2^2)Lc^3}{4E\hbar}\right)$$

This implies the probability of detecting the particle as being a muon neutrino is maximum when

$$\frac{(m_1^2 - m_2^2)Lc^3}{4E\hbar} = \pi \cdot n + \frac{\pi}{2}, \qquad n = 1, 2, 3, \ldots$$

According to the text,

$$m_1^2 - m_2^2 \approx 8 \times 10^{-5}\text{eV}^2/c^4$$

How far must it travel before it ocillates to a neutrino? THe shortest distance it must travel before it hits its maximum muon-neutrino likelyness is the length that satisfies

$$\frac{\pi}{2} = \frac{(m_1^2 - m_2^2)Lc^3}{4E\hbar}$$
$$L = \frac{4E\hbar\pi}{2(m_1^2 - m_2^2)c^3}$$
$$L = \frac{2E\hbar\pi}{\frac{(8\times10^{-5}\text{eV}^2)}{c^4}c^3}$$
$$L = \frac{2E\hbar\pi c}{(8\times 10^{-5}\text{eV}^2)}$$
$$L = \frac{(2)(8\text{MeV})(1.0545718\times 10^{-34}\text{m}^2\text{Kg/s})(\pi)(2.99792458\times 10^8\,\frac{\text{m}}{\text{s}})}{(8\times 10^{-5}\text{eV}^2)}$$
$$L = 19.844582\text{fm}$$

How many ocillations will take place from the sun to earth? It can be seen that the distance for one ocillation is going to be double the distance for a half an ocillation. The abouve calculated a havlf ocillation. Therefore there is one neutrino ocillation for every $\omega = 2\cdot 19.844582\text{fm} = 39.72891648\text{fm}$.The distance from the earth to the sun is $AU = 1.49597870700\times 10^{11}\text{m}$. The number of ocillations from the earth to the sun can be calculated by

$$\frac{AU}{\omega} = \frac{1.49597870700\times 10^{11}\text{m}}{39.72891648\text{fm}} = 5.032066029\times 10^{24}$$

This is about 5 septillion ocillations.

How many ocillations through the earth? The earth has a diameter of $d = 1.2756273\times 10^7\text{m}$.

$$\frac{d}{\omega} = 1.2756273\times 10^7\text{m}39.72891648\text{fm} = 3.210828316\times 10^{20}$$

This is about 321 quintillion ocillations.